Zadania powtórkowe

1. Obliczyć granicę:

   a) $\lim\limits_{n\to\infty} \left(\sqrt{3n^2+2n-5} - n\sqrt{3}\right)$

   b) $\lim\limits_{n\to\infty} \left(\sqrt{4n^2} - 2n\right)$

   c) $\lim\limits_{n\to\infty} \left(\sqrt{5}n - \sqrt{5n^2-7}\right)$

   d) $\lim\limits_{n\to\infty} \left(1+\frac{1}{n+2}\right)^{3n}$

   e) $\lim\limits_{n\to\infty} \left(1-\frac{1}{n}\right)^{n}$

   f) $\lim\limits_{n\to\infty} \left(\frac{4n}{4n+1}\right)^{n}$

   g) $\lim\limits_{n\to\infty} \left(\frac{n^2+6}{n^2}\right)^{n^2}$

   h) $\lim\limits_{n\to\infty} \left(\frac{n^2+4n}{n^2}\right)^{12n}$

   i) $\lim\limits_{n\to\infty} \left(\frac{n^2-3}{n^2}\right)^{3-2n^2}$

2. Zbadać ciągłość funkcji f, gdy:

   a)

$$f(x) = \begin{cases} x^2 & \text{dla } x \in \langle 0,1\rangle \\ 2-x^2 & \text{dla } x \in (1,2\rangle \end{cases}$$

   b)

$$f(x) = \begin{cases} x-1 & \text{dla } x < 1 \\ \log x & \text{dla } x \geqslant 1 \end{cases}$$

   c)

$$f(x) = \begin{cases} \dfrac{x^2-25}{x+5} & \text{dla } x \neq -5 \\ -10 & \text{dla } x = -5 \end{cases}$$

   d)

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{dla } x \neq 0 \\ 1 & \text{dla } x = 0 \end{cases}$$

   e)

$$f(x) = \begin{cases} x+1 & \text{dla } x < 0 \\ 0 & \text{dla } x = 0 \\ -x+1 & \text{dla} x > 0 \end{cases}$$

   f)

$$f(x) = \begin{cases} -\dfrac{x^3-x^2}{x-1} & \text{dla } x \in \langle -3,1) \\ 1 & \text{dla } x = 1 \\ \dfrac{x^3-x^2}{x-1} & \text{dla} x \in (1,2\rangle \end{cases}$$

3. Dobrać tak parametr a, aby funkcja f była ciągła na zbiorze liczb rzeczywistych:

a)

$$f(x) = \begin{cases} 2^x + 8 & \text{dla } x \leqslant 0 \\ (x-a)^2 & \text{dla } x > 0 \end{cases}$$

b)

$$f(x) = \begin{cases} \dfrac{\sqrt{1+x}-1}{x} & \text{dla } x \neq 0 \\ a & \text{dla } x = 0 \end{cases}$$

c)

$$f(x) = \begin{cases} -2\sin x & \text{dla } x < -\frac{\pi}{2} \\ a\sin x + 1 & \text{dla } x \in \langle -\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$$

4. Zbadać monotoniczność oraz wyznaczyć ekstrema lokalne funkcji

a) $f(x) = 2x^3 - 15x^2 + 36x - 14$

b) $f(x) = 4x + \dfrac{1}{x}$

c) $f(x) = x^2 e^{-x}$

d) $f(x) = \dfrac{1}{x} + 5\mathrm{arctg}x$

e) $f(x) = \dfrac{3x-2}{x+3}$

f) $f(x) = \dfrac{x^2+3}{x+1}$

g) $f(x) = \dfrac{x^2+2x}{x^2-4}$

h) $f(x) = \frac{6}{x} + \frac{1}{2}x^2 + 5x$

i) $f(x) = 2xe^x - 8e^x - x^2 + 6x$

j) $f(x) = \ln(x+1) - \mathrm{arctg}x$

k) $f(x) = \dfrac{x}{\ln x}$

ODP:

a) $\uparrow (-\infty, 2), (3, +\infty); \downarrow (2, 3)$; max: $x = 2$, min: $x = 3$

b) $\uparrow (-\infty, -\frac{1}{2}), (\frac{1}{2}, +\infty); \downarrow (-\frac{1}{2}, 0), (0, \frac{1}{2})$; max: $x = -\frac{1}{2}$, min: $x = \frac{1}{2}$

c) $\uparrow (0, 2), \downarrow (-\infty, 0), (2, +\infty)$, max: $x = 2$, min: $x = 0$

d) $\uparrow (-\infty, -\frac{1}{2}), (\frac{1}{2}, +\infty); \downarrow (-\frac{1}{2}, 0), (0, \frac{1}{2})$; max: $x = \frac{1}{2}$, min: $x = -\frac{1}{2}$

e) $\uparrow (-\infty, -3), (-3, +\infty)$ brak ekstremów;

f) $\uparrow (-\infty, -3), (1, +\infty); \downarrow (-3, 1)$; max: $x = -3$, min: $x = 1$

g) $\downarrow (-\infty, -2), (-2, 2)$; brak ekstremów

h) $\uparrow (-\infty, -3-\sqrt{3}), (-3+\sqrt{3}, 0), (0, 1); \downarrow (-3-\sqrt{3}, -3+\sqrt{3}), (1, +\infty)$; max: $x = -3-\sqrt{3}, x = 1$, min: $x = -3 + \sqrt{3}$

i) $\uparrow (-\infty, 0), (3, +\infty); \downarrow (0, 3)$; max: $x = 0$, min: $x = 3$

j) $\uparrow (-1, 0), (1, +\infty); \downarrow (0, 1)$; max: $x = 0$, min: $x = 1$

k) $\uparrow (e, +\infty); \downarrow (0, 1), (1, e)$; min: $x = e$

5. Określić przedziały wypukłości oraz wyznaczyć punkty przegięcia podanych funkcji

a) $f(x) = x^4 - 6x^2 - 6x + 1$

b) $f(x) = \dfrac{x^2}{(x-1)^3}$

c) $f(x) = xe^{-\frac{1}{2}x^2}$

d) $f(x) = e^{-x^3}$

e) $f(x) = \dfrac{x+7}{x^2+5}$

f) $f(x) = \dfrac{x^4}{x^3-1}$

g) $f(x) = \dfrac{7x}{x^2+1}$

h) $f(x) = \dfrac{x+1}{x^2+1}$

i) $f(x) = \dfrac{x}{2+x^2}$

j) $f(x) = x^2 + \ln(x+1)$

k) $f(x) = \dfrac{\ln x}{\sqrt{x}}$

l) $f(x) = x^2 e^{-0.5x^2}$

ODP

a) punkty przegięcia: $P_1 = (-1, 2), P_2 = (1, -10)$

b) punkty przegięcia: $P_1 = \left(-2-\sqrt{3}, \dfrac{-7-4\sqrt{3}}{54+30\sqrt{3}}\right)$, $P_2 = \left(-2+\sqrt{3}, \dfrac{7-4\sqrt{3}}{15\sqrt{3}-26}\right)$

c) punkty przegięcia: $P_1 = (-\sqrt{3}, -\sqrt{3}e^{-\frac{3}{2}})$, $P_2 = (0,0)$, $P_3 = (\sqrt{3}, \sqrt{3}e^{-\frac{3}{2}})$

d) punkty przegięcia: $P_1 = (0,1)$, $P_2 = (\sqrt[3]{\frac{2}{3}}, e^{-\frac{2}{3}})$

e) punkty przegięcia: $P_1 = (3\sqrt{15}-10, \frac{11+3\sqrt{15}}{20})$, $P_2 = (-1,1)$, $P_3 = (-3-\sqrt{15}-10, \frac{11-3\sqrt{15}}{20})$

f) punkty przegięcia: $P_1 = (-\sqrt[3]{2}, -\frac{2\sqrt[3]{2}}{3})$,

g) punkty przegięcia: $P_1 = (-\sqrt{3}, -\frac{7\sqrt{3}}{4})$, $P_2 = (0,0)$, $P_3 = (\sqrt{3}, \frac{7\sqrt{3}}{4})$

h) punkty przegięcia: $P_1 = (-2-\sqrt{3}, \frac{1-\sqrt{3}}{4})$, $P_2 = (\sqrt{3}-2, \frac{1+\sqrt{3}}{4})$, $P_3 = (1,1)$

i) punkty przegięcia: $P_1 = (-\sqrt{6}, -\frac{\sqrt{6}}{8})$, $P_2 = (0,0)$, $P_3 = (\sqrt{6}, \frac{\sqrt{6}}{8})$

j) punkty przegięcia: $P_1 = (\frac{\sqrt{2}}{2}-1, \frac{3}{2}-\sqrt{2}-\frac{1}{2}\ln 2)$

k) punkt przegięcia $P = (e^{\frac{8}{3}}; \frac{8}{3}e^{-\frac{4}{3}})$

l) punkty przegięcia $P_1 = (-\sqrt{3}; -\sqrt{3}e^{-\frac{3}{2}}); P_2 = (0,0); P_3 = (\sqrt{3}; \sqrt{3}e^{-\frac{3}{2}})$

6. Obliczyć granicę, korzystając z twierdzenia de L'Hospitala

a) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+6x}-1}{x} =$

b) $\lim\limits_{x\to 0^+} \dfrac{\ln\cos x}{\ln\cos 2x} =$

c) $\lim\limits_{x\to +\infty} \dfrac{\sqrt{6x-11}}{\ln x} =$

d) $\lim\limits_{x\to +\infty} x^2 e^{-x} =$

e) $\lim\limits_{x\to \frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x}$

f) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x}\right)^{\sin x}$

g) $\lim\limits_{x\to \pi} \left(\dfrac{1}{\sin x} - \dfrac{1}{\pi - x}\right)$

h) $\lim\limits_{x\to 0} \dfrac{\sin 3x - 3x}{x^3} =$

i) $\lim_{x\to 0} \dfrac{\cos 4x - 1}{7x^2}$

j) $\lim_{x\to 0} \dfrac{2\cos x - 2 + x^2}{x^4}$

k) $\lim_{x\to 0} \dfrac{e^{2x} - e^{-2x} - 4x}{x - \sin x}$

ODP:

a) 2

b) $\frac{1}{4}$

c) $+\infty$

d) 0

e) -1

f) 1

g) 0

h) $-\dfrac{9}{2}$

i) $-\dfrac{8}{7}$

j) $\dfrac{1}{12}$

k) 16

7. Obliczyć całki:

a) $\int(5x^2 - 6x + 3 - \frac{2}{x} + \frac{5}{x^2})\mathrm{d}x = \frac{5}{3}x^3 - 3x^2 - 3x - \ln|x| + \frac{5}{x} + C,$
$\int\left(2x - 3x^2 + \frac{1}{2\sqrt{x}}\right)\mathrm{d}x = x^2 - x^3 + \sqrt{x} + C,$

b) $\int(\sin x - \cos x)\mathrm{d}x = -\cos x - \sin x + C,$
$\int\left(e^{3x} + e^{\frac{x}{3}}\right)\mathrm{d}x = \frac{1}{3}e^{3x} + 3e^{\frac{x}{3}} + C,$

c) $\int \dfrac{\mathrm{d}x}{\sin^2 x \cos^2 x} = \operatorname{tg} x - \operatorname{ctg} x + C,$
$\int \operatorname{ctg}^2 x\mathrm{d}x = -\operatorname{ctg} x - x + C,$

d) $\int \dfrac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln 1 + x^2 + C,$
$\int \dfrac{x^2}{a^3 + x^3} = \frac{1}{3}\ln|a^3 + x^3| + C, a \neq 0,$

e) $\int \dfrac{x\sqrt[3]{x} + \sqrt[4]{x}}{x^2}\mathrm{d}x = 3x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{3}{4}} + C,$
$\int \dfrac{x}{\sqrt{x^2 - 6}}\mathrm{d}x = \sqrt{x^2 - 6} + C,$

f) $\int \dfrac{x}{\sqrt{3 - 5x^2}}\mathrm{d}x = -\frac{1}{5}\sqrt{3 - 5x^2} + C,$
$\int \dfrac{\cos x}{\sqrt{1 + \sin x}} = 2\sqrt{1 + \sin x} + C$

g) $\int \frac{e^x}{2e^x + 1}\mathrm{d}x = \frac{1}{2}\ln(2e^x + 1) + C$

8. Obliczyć całki metodą przez podstawienie:

a) $\int(5x^3 - 3)^7 x^2\mathrm{d}x = \frac{1}{120}(5x^3 - 3)^8 + C$
$\int(x^2 + 4)^5 x\mathrm{d}x = \frac{1}{12}(x^2 + 4)^6 + C$
$\int \sin^3 x\mathrm{d}x = \frac{1}{3}\cos^3 x - \cos x + C$

b) $\int \dfrac{x}{(x^2 + 3)^6}\mathrm{d}x = \dfrac{-1}{10(x^2 + 3)^5} + C$
$\int \dfrac{x}{\sqrt{1 - x^4}}\mathrm{d}x = \frac{1}{2}\arcsin x^2 + C$
$\int \dfrac{e^x}{e^{2x} + 1}\mathrm{d}x = \operatorname{arctg}(e^x) + C,$

c) $\int xe^{x^2}\mathrm{d}x = \frac{1}{2}e^{x^2} + C$
$\int \frac{\operatorname{tg} x}{(1+\operatorname{tg}^4 x)\cos^2 x}\mathrm{d}x = \frac{1}{2}\operatorname{arctg}(\operatorname{tg}^2 x) + C$
$\int \frac{x^7}{\sqrt{1-x^{16}}}\mathrm{d}x = \frac{1}{8}\arcsin(x^8) + C,$

d) $\int \frac{\sin^3 x}{\cos^{10} x}\mathrm{d}x = \frac{1}{9\cos^9 x} - \frac{1}{7\cos^7 x} + C$
$\int \frac{\ln x}{x}\mathrm{d}x = \frac{1}{2}\ln^2 x + C$
$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\mathrm{d}x = 2\sin\sqrt{x} + C,$

e) $\int \sqrt{3x+1}\mathrm{d}x = x \geqslant -\frac{1}{3} : \frac{2}{9}(3x+1)^{\frac{3}{2}} + C$
$\int \frac{x}{\sqrt[3]{2x^2-1}}\mathrm{d}x = \frac{3}{8}(2x^2-1)^{\frac{2}{3}}, x \neq \pm\frac{\sqrt{2}}{2}$
$\int \frac{x^2}{\cos^2(x^3+1)}\mathrm{d}x = \frac{1}{3}\operatorname{tg}(x^3+1) + C, \cos(x^3+1) \neq 0$

f) $\int x\sqrt{1+x^2}\mathrm{d}x = \frac{1}{3}\sqrt{(1+x^2)^3} + C$
$\int \frac{x^2}{\sqrt[5]{x^3+1}}\mathrm{d}x = \frac{5}{12}\sqrt[5]{(x^3+1)^4} + C$
$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\mathrm{d}x = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x}+1| + C,$

g) $\int \cos x e^{\sin x}\mathrm{d}x = e^{\sin x} + C$
$\int \frac{x^3}{\cos^2 x^4}\mathrm{d}x = \frac{1}{4}\operatorname{tg} x^4, \cos x^4 \neq 0$
$\int \frac{\cos x}{\sin^2 x + 4}\mathrm{d}x = \frac{1}{2}\operatorname{arctg}(\frac{1}{2}\sin x) + C,$

h) $\int \frac{(\ln x)^2}{x}\mathrm{d}x = \frac{1}{3}\ln^3 x + C$
$\int \frac{1}{e^x + e^{-x}}\mathrm{d}x = \operatorname{arctg} e^x + C$
$\int \sin^5 x\cos x\mathrm{d}x = \frac{1}{6}\sin^6 x + C,$

i) $\int \frac{\operatorname{tg} x}{\cos^2 x}\mathrm{d}x = \frac{1}{2}\operatorname{tg}^2 x + C, \cos x \neq 0$
$\int \sin x\cos^2 x\mathrm{d}x = -\frac{1}{3}\cos^3 x + C$
$\int \frac{\mathrm{d}x}{x(1+\ln x)} = \ln|\ln x + 1| + C$

9. Obliczyć całki metodą całkowania przez części

a) $\int \sqrt{x}\ln x\mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}}(\ln x - \frac{2}{3}) + C$
$\int \ln x\mathrm{d}x = x(\ln x - 1) + C$
$\int \ln(x^2+1)\mathrm{d}x = x\ln(x^2+1) - 2x + 2\operatorname{arctg} x + C,$

b) $\int x^2e^x\mathrm{d}x = (x^2-2x+2)e^x + C$
$\int x\cos x\mathrm{d}x = x\sin x + \cos x + C$
$\int x\cos 3x\mathrm{d}x = \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C,$

c) $\int \frac{\ln x}{x^5}\mathrm{d}x = -\frac{1}{4x^4}(\ln x + \frac{1}{4}) + C$
$\int e^{-2x}\sin 3x\mathrm{d}x = -\frac{1}{13}e^{-2x}(2\sin 3x + 3\cos 3x) + C$
$\int e^x\cos x\mathrm{d}x = \frac{1}{2}e^x(\sin x + \cos x) + C,$

d) $\int \arcsin x\mathrm{d}x = x\arcsin x + \sqrt{1-x^2} + C$
$\int x^2e^{-x}\mathrm{d}x = -e^{-x}(2x + x^2 + 2) + C$
$\int x\sin 2x\mathrm{d}x = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C,$

e) $\int \sin\sqrt{x}\mathrm{d}x = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$
$\int x^3e^{-x^2}\mathrm{d}x = \frac{e^{-x^2}(x^2+1)}{2} + C$
$\int \frac{\ln x - 1}{\ln^2 x}\mathrm{d}x = \frac{x}{\ln x} + C$

f) $\int \arcsin x \mathrm{d}x = x \arcsin x + \sqrt{1-x^2} + C$
$\int \operatorname{arc\,tg} x \mathrm{d}x = x \operatorname{arc\,tg} x - \frac{1}{2} \ln |1+x^2| + C$
$\int \ln x \mathrm{d}x = x \ln |x| - x + C$

10. Obliczyć:

a) $\int \frac{\mathrm{d}x}{(3x-2)^4} = \frac{-1}{9(3x-2)^3} + C, x \neq \frac{2}{3},$
$\int \frac{\mathrm{d}x}{(x-2)^2} = \frac{-1}{x-2} + C, x \neq 2$
$\int \frac{3x-4}{x^2-x-6} \mathrm{d}x = \ln|x-3| + 2\ln|x+2| + C, x \neq 3 \wedge x \neq -2$

b) $\int \frac{2x-3}{x^2-3x+3} \mathrm{d}x = \ln|x^2-3x+3| + C$
$\int \frac{x+13}{x^2-4x-5} \mathrm{d}x = 3\ln|x-5| - 2\ln|x+1| + C, x \neq 5 \wedge x \neq -1$
$\int \frac{6x-13}{x^2-\frac{7}{2}x+\frac{3}{2}} \mathrm{d}x = 2\ln|x-3| + 4\ln|x-\frac{1}{2}| + C, x \neq 3 \wedge x \neq \frac{1}{2}$

c) $\int \frac{2x+6}{2x^2+3x+1} \mathrm{d}x = 5\ln|x+\frac{1}{2}| - 4\ln|x+1| + C, x \neq -\frac{1}{2} \wedge x \neq -1$
$\int \frac{\frac{5}{6}x-16}{x^2+3x-18} \mathrm{d}x = \frac{7}{3}\ln|x+6| - \frac{3}{2}\ln|x-3| + C, x \neq -6 \wedge x \neq 3$
$\int \frac{5+x}{10x+x^2} \mathrm{d}x = \frac{1}{2}\ln|10x+x^2| + C$

d) $\int \frac{\mathrm{d}x}{x^2+2x-1} = \frac{\sqrt{2}}{4} \ln\left|\frac{x+1-\sqrt{2}}{x+1+\sqrt{2}}\right| + C$
$\int \frac{\mathrm{d}x}{6x^2-13x+6} = \ln\left|\frac{2x-3}{3x-2}\right| + C, x \neq \frac{2}{3} \wedge x \neq \frac{3}{2}$
$\int \frac{7x}{4+5x^2} \mathrm{d}x = \frac{7}{10}\ln|4+5x^2| + C$

e) $\int \frac{2x-13}{(x-5)^2} \mathrm{d}x = 2\ln|x-5| + \frac{3}{x-5} + C, x \neq 5$
$\int \frac{3x+1}{(x+2)^2} \mathrm{d}x = 3\ln|x+2| + \frac{5}{x+2}, x \neq -2$

11. Obliczyć:

a) $\int \frac{\mathrm{d}x}{2x^2-2x+5} = \frac{1}{3}\operatorname{arctg}\frac{2x-1}{3} + C$
$\int \frac{\mathrm{d}x}{3x^2+2x+1} = \frac{\sqrt{2}}{2}\operatorname{arctg}\frac{3x+1}{\sqrt{2}} + C$
$\int \frac{\mathrm{d}x}{13-6x+x^2} = \frac{1}{2}\operatorname{arctg}\frac{x-3}{2} + C$

b) $\int \frac{4x-1}{2x^2-2x+1} \mathrm{d}x = \ln|2x^2-2x+1| + \operatorname{arctg}(2x-1) + C$
$\int \frac{2x-1}{x^2-2x+5} \mathrm{d}x = \ln|x^2-2x+5| + \frac{1}{2}\operatorname{arctg}\frac{x-1}{2} + C$
$\int \frac{\mathrm{d}x}{9x^2-6x+2} = \operatorname{arctg}(3x-1) + C$

c) $\int \frac{2x-20}{x^2-8x+25} \mathrm{d}x = \ln|x^2-8x+25| - 4\operatorname{arctg}\frac{x-4}{3} + C$
$\int \frac{3x+4}{x^2+4x+8} \mathrm{d}x = \frac{3}{2}\ln|x^2+4x+8| - \operatorname{arctg}\frac{x+2}{2} + C$
$\int \frac{x-6}{x^2-3} \mathrm{d}x = \frac{1}{2}\ln|x^2-3| + \sqrt{3}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C$

d) $\int \frac{10x-44}{x^2-4x+20} \mathrm{d}x = 5\ln|x^2-4x+20| + 6\operatorname{arctg}\frac{x-2}{4} + C$
$\int \frac{4x-5}{x^2-6x+10} \mathrm{d}x = 2\ln|x^2-6x+10| + 7\operatorname{arctg}(x-3) + C$
$\int \frac{x+6}{x^2+3} \mathrm{d}x = \frac{1}{2}\ln|x^2+3| + 2\sqrt{3}\operatorname{arctg}\frac{x}{\sqrt{3}} + C$