1 MM-ZI, Matematyka 1

Przykłady: potęgowanie i pierwiastkowanie liczb zespolonych

Przykład 1. Obliczyć $(-2\sqrt{3}+2i)^8$.

Rozwiązanie: Przypomnienie: $z^n = |z|^n(\cos n\alpha + i\sin n\alpha)$.

Wiemy, że $z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$.

$$z^8 = 4^8\left(\cos\left(8\cdot\frac{5\pi}{6}\right) + i\sin\left(8\cdot\frac{5\pi}{6}\right)\right)$$
$$z^8 = 4^8\left(\cos\frac{20\pi}{3} + i\sin\frac{20\pi}{3}\right)$$
$$z^8 = 4^8\Big(\cos\left(6\frac{2}{3}\pi\right) + i\sin\left(6\frac{2}{3}\pi\right)\Big)$$
$$z^8 = 4^8\Big(\cos\left(\frac{2}{3}\pi + 3\cdot 2\pi\right) + i\sin\left(\frac{2}{3}\pi + 3\cdot 2\pi\right)\Big)$$
$$z^8 = \left(2^2\right)^8\left(\cos\left(\pi-\frac{\pi}{3}\right) + i\sin\left(\pi-\frac{\pi}{3}\right)\right)$$
$$z^8 = 2^{16}\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$z^8 = 2^{16}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$z^8 = 2^{15}(-1+\sqrt{3}i)$$

Przykład 2. Obliczyć $\sqrt[4]{-8-8\sqrt{3}i}$

Rozwiązanie: Przypomnienie: $\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\alpha+2k\pi}{n} + i\sin\frac{\alpha+2k\pi}{n}\right)$, $k = 0, 1, \dots, n-1$.

Przedstawmy liczbę $z = -8-8\sqrt{3}i$ w postaci trygonometrycznej. Mamy

$|z| = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = \sqrt{64+3\cdot 64} = \sqrt{4\cdot 64} = 2\cdot 8 = 16.$

Stąd $\begin{cases} \cos\alpha = \dfrac{-8}{16} \\ \sin\alpha = \dfrac{-8\sqrt{3}}{16} \end{cases} \Leftrightarrow \begin{cases} \cos\alpha = -\frac{1}{2} \\ \sin\alpha = -\frac{\sqrt{3}}{2} \end{cases} \Leftrightarrow \alpha = \pi + \frac{\pi}{3} = \frac{4\pi}{3}.$

Zatem $z = 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ jest postacią trygonometryczną liczby $z = -8-8\sqrt{3}i$.

Obliczamy

$$\sqrt[4]{-8-8\sqrt{3}i} = \sqrt[4]{16}\left(\cos\frac{\frac{4\pi}{3}+2k\pi}{4} + i\sin\frac{\frac{4\pi}{3}+2k\pi}{4}\right), k = 0,1,2,3.$$
$$\sqrt[4]{-8-8\sqrt{3}i} = 2\left(\cos\frac{\frac{4\pi}{3}+2k\pi}{4} + i\sin\frac{\frac{4\pi}{3}+2k\pi}{4}\right), k = 0,1,2,3.$$

Gdy $k = 0$ mamy

$$w_0 = 2\left(\cos\frac{\frac{4\pi}{3} + 2\cdot 0\cdot \pi}{4} + i\sin\frac{\frac{4\pi}{3} + 2\cdot 0\cdot \pi}{4}\right)$$
$$w_0 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$w_0 = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$w_0 = 1 + \sqrt{3}i$$

Gdy $k = 1$ mamy
$$w_1 = 2\left(\cos\frac{\frac{4\pi}{3} + 2\cdot 1\cdot \pi}{4} + i\sin\frac{\frac{4\pi}{3} + 2\cdot 1\cdot \pi}{4}\right)$$
$$w_1 = 2\left(\cos\frac{10\pi}{12} + i\sin\frac{10\pi}{12}\right)$$

$$w_1 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$
$$w_1 = 2\left(\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right)$$
$$w_1 = 2\left(-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$w_1 = 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$w_1 = -\sqrt{3} + i$$

Gdy $k = 2$ mamy
$$w_2 = 2\left(\cos\frac{\frac{4\pi}{3} + 2\cdot 2\cdot \pi}{4} + i\sin\frac{\frac{4\pi}{3} + 2\cdot 2\cdot \pi}{4}\right)$$
$$w_2 = 2\left(\cos\frac{16\pi}{12} + i\sin\frac{16\pi}{12}\right)$$
$$w_2 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$
$$w_2 = 2\left(\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right)$$
$$w_2 = 2\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$
$$w_2 = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$w_2 = -1 - \sqrt{3}i$$

Gdy $k = 3$ mamy

$$w_3 = 2\left(\cos\frac{\frac{4\pi}{3} + 2\cdot 3\cdot\pi}{4} + i\sin\frac{\frac{4\pi}{3} + 2\cdot 3\cdot\pi}{4}\right)$$
$$w_3 = 2\left(\cos\frac{22\pi}{12} + i\sin\frac{22\pi}{12}\right)$$
$$w_3 = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$
$$w_3 = 2\left(\cos\left(2\pi - \frac{\pi}{6}\right) + i\sin\left(2\pi - \frac{\pi}{6}\right)\right)$$
$$w_3 = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$
$$w_3 = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$
$$w_3 = \sqrt{3} - i$$